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Technical Notes

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Nondestructive Determination of Additional Buckling Load for Preloaded Plates and Bars

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Introduction

THE generic integral equations for nondestructive determination of the buckling loads obtained previously can be used to obtain the additional load that would cause the preloaded structure to buckle. The integral equations for nondestructive determination of buckling loads for elastic plates and bars have been derived, for example, in [1,2] (in which one can find many references connected with the discussed subject). They have been used for nondestructive determination of elastic columns [2,3], elastic flat plates [4], and damaged composite plates [5]. In [4] appears a very interesting sentence: "Equation (3) [Eq. (22) of [2]] is linear with respect to the applied loads. Hence, any preload may exist, and the calculated buckling load will then be the load above the preload." In fact, the aforementioned equation is nonlinear in respect to the axial load. However, interestingly enough, the conclusion of [4] is basically right. More than this, the preload and the additional loads that cause the structure to buckle may even have different topological configurations.

Generic Integral Equation for Buckling Load of Preloaded Columns

A Timoshenko-type column (Fig. 1) is constrained on the boundaries by coupled elastic springs. The column is preloaded by an axial force $N_0(s)$. It is postulated that $N_0(s)$ does not change during the loading. The additional axial force $N(s)$ causes the column to buckle. The springs are defined by the spring constant positive definite symmetric matrices [6] $\alpha(2\cdot 2)$ and $\beta(2\cdot 2)$ for $s=0$, 1, respectively. Axial forces load the column. The stability equations and the boundary conditions will be obtained by using the principle of minimum additional potential energy during buckling [7]. Hence,

$$\delta(U+V)=0 \tag{1}$$

where U is the strain energy and V is the potential energy of the external load.

$$\begin{aligned} V &= -\frac{1}{2}\int_0^l\left[(P_{p1}+P_1)+\int_0^s[p_0(\eta)+p(\eta)]\,\mathrm{d}\eta\right]\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2\mathrm{d}s \\ &= -\frac{1}{2}\int_0^l[N_0(s)+N(s)]\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2\mathrm{d}s \end{aligned}\tag{2}$$

The span of the beam is l, $N_0(s)$ is the varying preload, and $N(s)$ is the varying axial force that will cause the beam to buckle. For N_0 and N, compression is assumed to be positive [1]. Note that N_0 and N may have different topological shapes. Hence,

$$\delta(U+V)=\delta U-\int_0^l(N_0+N)\delta\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)\mathrm{d}s=0 \tag{3}$$

The strain energy for a Timoshenko beam is given by [1]

$$\begin{aligned} U &= \frac{1}{2}\int_0^l EI\left(\frac{\mathrm{d}\psi}{\mathrm{d}s}\right)^2\mathrm{d}s+\frac{1}{2}\int_0^l kGA\left(\frac{\mathrm{d}y}{\mathrm{d}s}-\psi\right)^2\mathrm{d}s \\ &\quad+\frac{1}{2}J_0^t\alpha J_0+\frac{1}{2}J_l^t\beta J_1 \end{aligned}\tag{4}$$

where E is the varying modulus of elasticity, I is the varying moment of inertia, ψ is the rotation of the cross section of the beam, k is the varying shear factor, G is the varying shear modulus, A is the varying cross section, and y is its deflection. It must be emphasized that the Green function $v(x,s)$ is obtained from experiments and the evaluation of the shear factor k is not needed. J is given as follows:

$$J=\begin{Bmatrix}\psi\\ y\end{Bmatrix} \tag{5}$$

By performing the variations, one obtains

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\left[kGA\left(\frac{\mathrm{d}y}{\mathrm{d}s}-\psi\right)\right]-\frac{\mathrm{d}}{\mathrm{d}s}\left(N_0\frac{\mathrm{d}y}{\mathrm{d}s}\right)-\frac{\mathrm{d}}{\mathrm{d}s}\left(N\frac{\mathrm{d}y}{\mathrm{d}s}\right)=0 \\ &\frac{\mathrm{d}}{\mathrm{d}s}\left(EI\frac{\mathrm{d}\psi}{\mathrm{d}s}\right)+kGA\left(\frac{\mathrm{d}y}{\mathrm{d}s}-\psi\right)=0 \end{aligned}\tag{6}$$

with the boundary conditions, for $s=0$,

$$\begin{aligned} &-EI\frac{\mathrm{d}\psi}{\mathrm{d}s}+\alpha_{11}\psi+\alpha_{12}y=0 \quad\text{or}\quad \psi=0 \\ &-kGA\left(\frac{\mathrm{d}y}{\mathrm{d}s}-\psi\right)+\alpha_{21}\psi+\alpha_{22}y+(N_0+N)\frac{\mathrm{d}y}{\mathrm{d}s}=0 \quad\text{or}\quad y=0 \end{aligned}\tag{7}$$

and for $s=l$,

$$\begin{aligned} &EI\frac{\mathrm{d}\psi}{\mathrm{d}s}+\beta_{11}\psi+\beta_{12}y=0 \quad\text{or}\quad \psi=0 \\ &kGA\left(\frac{\mathrm{d}y}{\mathrm{d}s}-\psi\right)+\beta_{21}\psi+\beta_{22}y-(N_0+N)\frac{\mathrm{d}y}{\mathrm{d}s}=0 \quad\text{or}\quad y=0 \end{aligned}\tag{8}$$

Note that the eigenvalue N appears, as expected, in the boundary conditions as well as in the stability equations.

We will examine now the behavior of a preloaded beam loaded by forces perpendicular to its axis (see Fig. 2).

Using the approach by which Eqs. (6–8) were derived, one can obtain the equilibrium equations for the preloaded beam of Fig. 2:

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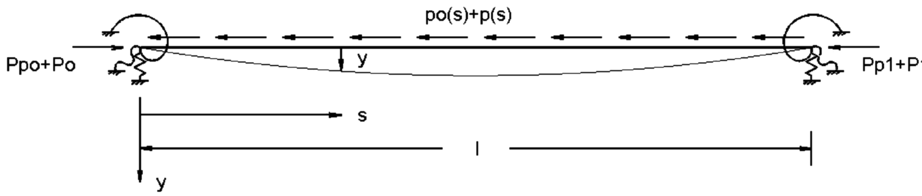

Fig. 1 Axially preloaded, elastically constrained beam loaded by axial forces.

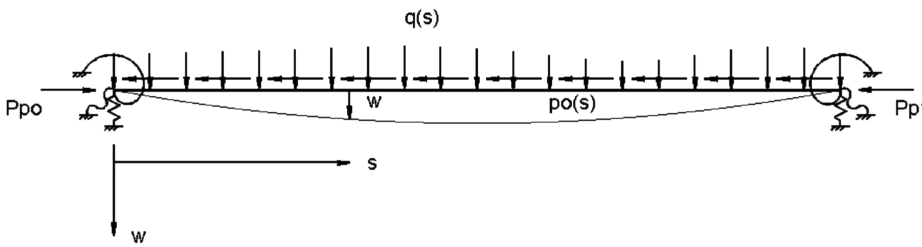

Fig. 2 Preloaded beam loaded by forces perpendicular to its axis.

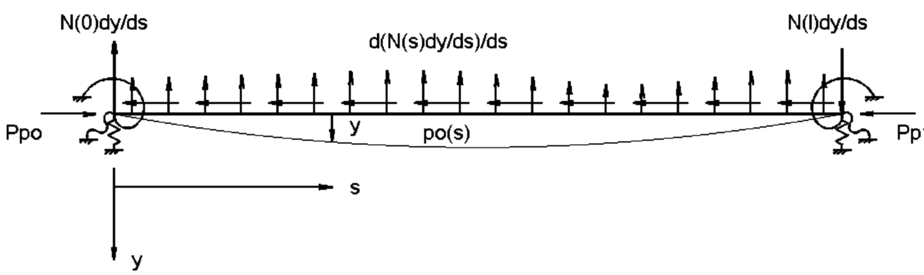

Fig. 3 Equivalent preloaded Timoshenko beam.

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\left[kGA\left(\frac{\mathrm{d}w}{\mathrm{d}s}-\psi_p\right)\right]-\frac{\mathrm{d}}{\mathrm{d}s}\left(N_0\frac{\mathrm{d}w}{\mathrm{d}s}\right)+q=0\\&\frac{\mathrm{d}}{\mathrm{d}s}\left(EI\frac{\mathrm{d}\psi_p}{\mathrm{d}s}\right)+kGA\left(\frac{\mathrm{d}w}{\mathrm{d}s}-\psi_p\right)=0\end{aligned}\tag{9}$$

with the appropriate boundary conditions [6]. The deflection of the preloaded beam is w, ψ_p is its rotation, and q is the load per unit length.

Equations (9) are nonlinear in respect to the preload N_0. However, N_0 is prescribed and does not change during application of the load q. N_0 can be visualized as some complicated elastic foundation. One can see that Eqs. (9) are linear in respect to loads perpendicular to the axis. Hence, the Maxwell–Betti reciprocal theorem [7,8] is applicable for such loads.

Now, the Timoshenko column loaded by additional axial forces N, which generally appear in the boundary as well, will be replaced by an equivalent preloaded Timoshenko beam loaded by fictitious external loads [6] as shown in Fig. 3. Note that the concentrated forces are external loads and not boundary conditions. The spring constants on the boundaries have the same values as in the case of the original column (Fig. 1). Note that in the case of the equivalent beam, the eigenvalue N does not appear in the boundary conditions (see Fig. 3).

It can be easily shown that the column of Fig. 1 and the beam of Fig. 3 are equivalent [6]. To find the equilibrium equations of the equivalent Timoshenko beam, one can use again the principle of minimum potential energy:

$$\begin{aligned}\delta(U+V)&=\delta U-\frac{1}{2}\int_0^i N_0\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2\mathrm{d}s\\&\quad-\left[N\frac{\mathrm{d}y}{\mathrm{d}s}\delta y|_0^l-\int_0^l\frac{\mathrm{d}}{\mathrm{d}s}\left(N\frac{\mathrm{d}y}{\mathrm{d}s}\right)\delta y\,\mathrm{d}s\right]\end{aligned}\tag{10}$$

It must be emphasized that the strain energy is exactly the same as in the case of the Timoshenko column. By performing the integration, one obtains Eq. (3), which is exactly the same one obtained for the preloaded column.

In the case of the equivalent Timoshenko beam (Fig. 3), the eigenvalue does not appear in the boundaries and, as shown earlier, the preloaded beam can be treated formally as a linear structure. In this case, the Maxwell–Betti reciprocal theorem [7,8] is applicable and will be used to obtain the kernel of the integral equation for the preloaded Timoshenko column.

The elastically restrained preloaded Timoshenko beam is loaded once by the fictitious loads (Fig. 3) and once by a unit force $F=1$ (Fig. 4).

By applying the Maxwell–Betti reciprocal theorem, one obtains

$$y(x)=N\frac{\mathrm{d}y}{\mathrm{d}s}v|_0^l-\int_0^l\frac{\mathrm{d}}{\mathrm{d}s}\left(N(s)\frac{\mathrm{d}y(s)}{\mathrm{d}s}\right)v(x,s)\,\mathrm{d}s\tag{11a}$$

$$y(x)=\int_0^l\frac{\partial v(x,s)}{\partial s}N(s)\frac{\mathrm{d}y(s)}{\mathrm{d}s}\,\mathrm{d}s\tag{11b}$$

where $y(x)$ is the deflection of the buckled column at point x due to the compressive forces (Fig. 1) and $v(x,s)$ is the deflection of the preloaded beam at point s due to a unit force at point x. By representation of $N(s)$ as

$$N(s)=\lambda n(s)\tag{12a}$$

where λ is the required eigenvalue and $n(s)$ is a known function, one obtains

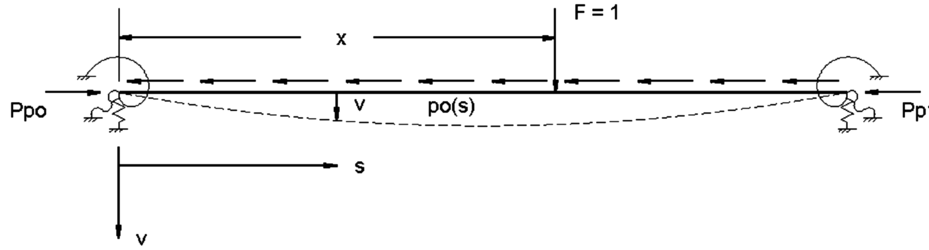

Fig. 4 Preloaded beam loaded by a unit force.

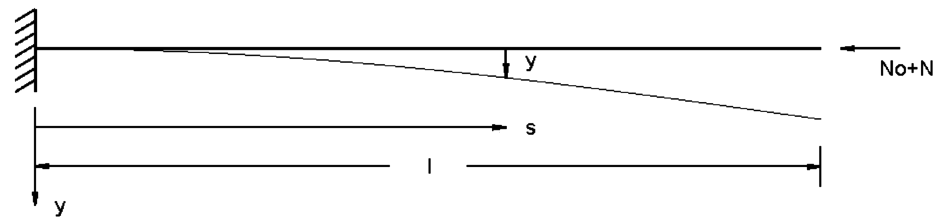

Fig. 5 Preloaded cantilever beam-column loaded by an additional constant axial force.

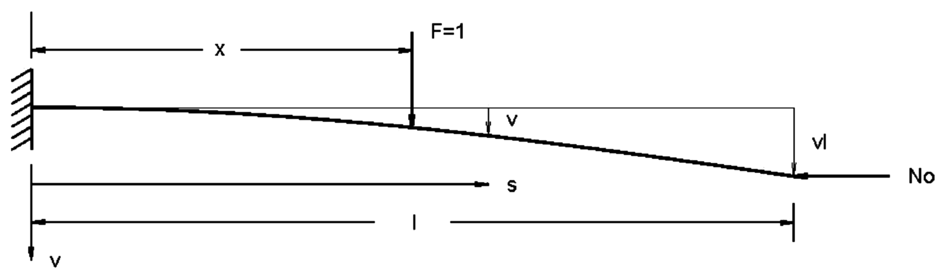

Fig. 6 Preloaded cantilever beam loaded by unit force.

$$y(s) = \lambda \int_0^l \frac{\partial v(x,s)}{\partial s} n(s) \frac{\mathrm{d}y(s)}{\mathrm{d}s}\, \mathrm{d}s \tag{12b}$$

Equation (11b), or its equivalent Eq. (12), is the required integral equation for the calculation of the additional load N that would cause the beam to buckle. It must be emphasized that Eq. (11b) was obtained for the Timoshenko beam, and hence it takes into account the influence of the shear strain on the magnitude of the buckling load. The kernel $v(x,s)$ is of course strongly influenced by the preload N_0. Note that $v(x,s)$ satisfies the boundary conditions and therefore their determination is superfluous. Once the kernel is known, one can use one of the powerful numerical techniques of the integral equations to calculate the buckling load [1–3,6]. To include the influence of the shearing strains on the magnitude of the buckling load, Eq. (12) must be used as it is [1,6].

Example 1. Preloaded Timoshenko Cantilever Beam-Column

A constant preloaded axial force N_0 and an additional axial force N loads a cantilever beam-column with constant EI and constant kAG (Fig. 5). In this case, $n(s) = 1$ and Eq. (12) becomes

$$y(x) = N \int_0^l \frac{\partial v(x,s)}{\partial s} \frac{\mathrm{d}y(s)}{\mathrm{d}s}\, \mathrm{d}s \tag{13}$$

where N is the additional critical load.

To calculate the kernel, Fig. 6, one must apply the basic constitutive expressions of the Timoshenko beam theory [6]. The kernel obtained in this way is given as follows:

$$\begin{aligned} s < x \rightarrow v = {}& \frac{kAG}{\alpha N_0 (kAG - N_0)} \Big[\sin(\alpha s) - \tan(\alpha l)\cos(\alpha s) \\ & + \tan(\alpha l)\cos(\alpha x)\cos(\alpha s) - \sin(\alpha x)\cos(\alpha s) \\ & - \frac{kAG - N_0}{kAG} \alpha s + \tan(\alpha l) - \tan(\alpha l)\cos(\alpha x) + \sin(\alpha x) \Big] \\ s > x \rightarrow v = {}& \frac{kAG}{\alpha N_0 (kAG - N_0)} \Big[\sin(\alpha x) - \tan(\alpha l)\cos(\alpha x) \\ & + \tan(\alpha l)\cos(\alpha s)\cos(\alpha x) - \sin(\alpha s)\cos(\alpha x) \\ & - \frac{kAG - N_0}{kAG} \alpha x + \tan(\alpha l) - \tan(\alpha l)\cos(\alpha s) + \sin(\alpha s) \Big] \end{aligned} \tag{14}$$

$$\alpha^2 = \frac{N_0}{EI} \frac{kAG}{(kAG - N_0)} \tag{15}$$

As expected, the kernel of the preloaded cantilever beam $v(x,s)$ is symmetric and hence fulfills the Maxwell–Betti reciprocal theorem [7,8].

The eigenfunction for this case is given by [6]

$$y(x) = 1 - \cos\left(\frac{\pi x}{2l}\right) \tag{16}$$

Substitution into Eq. (13) yields N_1 which causes the beam to buckle,

$$N_1 = \frac{EI(\pi/2l)^2}{1 + (EI/kGA)(\pi/2l)^2} - N_0 = N_{\mathrm{cr}1} - N_0 \tag{17}$$

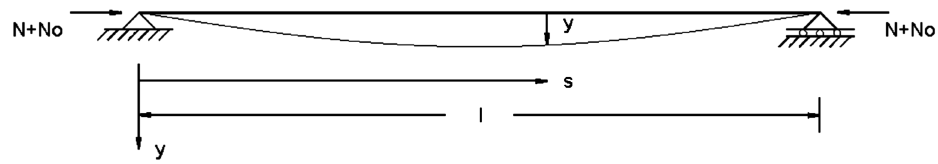

Fig. 7 Preloaded simply supported beam-column loaded by an additional constant force.

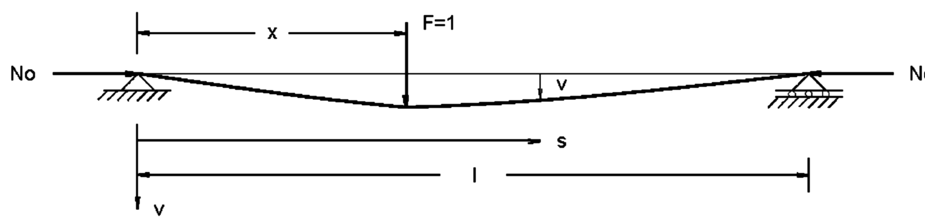

Fig. 8 Preloaded simply supported beam loaded by unit force.

where N_{cr1} is the critical axial load for a Timoshenko cantilever beam without preload [1].

Example 2. Simply Supported Preloaded Timoshenko Beam-Column

Constant axial forces $N + N_0$ load a simply supported beam-column with constant EI and constant kAG (Fig. 7). The kernel for this case (Fig. 8) is given by [6]

$$\begin{aligned} s < x \rightarrow v &= \frac{kAG}{\alpha N_0(kAG - N_0)}\left[\cos(\alpha x)\sin(\alpha s) - \frac{\sin(\alpha x)\sin(\alpha s)}{\tan(\alpha l)} - \frac{kAG - N_0}{kAG}\alpha\left(s - \frac{xs}{l}\right)\right] \\ s > x \rightarrow v &= \frac{kAG}{\alpha N_0(kAG - N_0)}\left[\cos(\alpha s)\sin(\alpha x) - \frac{\sin(\alpha s)\sin(\alpha x)}{\tan(\alpha l)} - \frac{kAG - N_0}{kAG}\alpha\left(x - \frac{sx}{l}\right)\right] \end{aligned} \tag{18}$$

As expected, the Green function $v(x, s)$ is again symmetric.

The eigenfunction for this case is given by [6]

$$y(x) = \sin\left(\frac{\pi x}{l}\right) \tag{19}$$

Substitution of Eq. (19) into Eq. (13) yields

$$N_2 = \frac{EI(\pi/l)^2}{1 + (EI/kAG)(\pi/l)^2} - N_0 = N_{cr2} - N_0 \tag{20}$$

Example 3. Adequate Approximation

It is interesting to show that Eqs. (17) and (20) can be obtained by using an approximation of the preloaded kernel $v(x, s)$. The approximation is a generalization of Timoshenko's approximation for the deflections of a preloaded beam-column [9]

$$\delta = \frac{\delta_0}{1 - (P/P_{cr})} \tag{21}$$

where δ is the deflection of the preloaded beam-column, δ_0 is the deflection of the beam without preload, P is the preload, and P_{cr} is the critical load of the beam. Equation (21) can be generalized to obtain an approximation of the Green function

$$v(x, s) = \frac{v_0(x, s)}{1 - (N_0/N_{cr})} \tag{22}$$

where $v_0(x, s)$ is the Green function of the beam without preload. Substitution of Eq. (22) into Eq. (13) yields

$$\begin{aligned} y(x) &= N\int_0^l \frac{\partial v(x,s)}{\partial s}\frac{\mathrm{d}y(s)}{\mathrm{d}s}\,\mathrm{d}s = \frac{N}{1-(N_0/N_{cr})}\int_0^l \frac{\partial v_0(x,s)}{\partial s}\frac{\mathrm{d}y(s)}{\mathrm{d}s}\,\mathrm{d}s \\ &= N_{cr}\int_0^l \frac{\partial v_0(x,s)}{\partial s}\frac{\mathrm{d}y(s)}{\mathrm{d}s}\,\mathrm{d}s \end{aligned} \tag{23a}$$

From Eq. (23a), one obtains

$$N = N_{cr} - N_0 \tag{23b}$$

which is the exact solution of the problem. At first glance it seems strange that the exact solution of Eq. (23b) was obtained by using the approximation of Eq. (22). However, the explanation is quite simple. Timoshenko [9] obtained the approximation of Eq. (21) by expanding the deflection of the preloaded beam-column using the infinite series of the buckling modes and taking only the fist mode. It will be shown in the next example that the integral equation awakens only one buckling mode. Hence, all the other modes become irrelevant.

Example 4. Buckling of Preloaded Plate

Following the approach for the preloaded Timoshenko beam given in the preceding section and the approach given in [1,2], it is clear that the generic integral equation for the buckling of the preloaded Mindlin–Reissner plate will be as the one obtained by using the classical Kirchhoff plate theory [2]. It must be emphasized that the Mindlin–Reissner plate theory is an extension of the Timoshenko beam theory. The generic integral equation for the buckling of plates can be found in [1,2,6] and will be not given here.

A special case arises when the preload constant forces N_x act in the x direction and the additional constant forces N_y in the y direction. In this case, the integral equation is given by [1,2,6] (Fig. 9)

$$w(x, y) = N_y \iint_A \frac{\partial R(x, y; s, t)}{\partial t}\frac{\partial w(s, t)}{\partial t}\,\mathrm{d}A \tag{24}$$

The Green function $R(x, y; s, t)$ of the preloaded plate is needed.

For simplicity, the calculations of example 4 will be presented by using Kirchhoff's plate theory [10] for a plate with constant stiffness D. The equation of equilibrium of the preloaded plate (Fig. 10) reads

$$D\nabla^4 R + N_x\frac{\partial^2 R}{\partial s^2} = q \tag{25}$$

The unit load $F(x, y) = 1$ on the plate was expanded by using the infinite series of the buckling modes connected with the simply supported plate [6]. The Green function was assumed to be

$$R = \sum_i \sum_j b_{ij}\sin\left(\frac{i\pi s}{a}\right)\sin\left(\frac{j\pi t}{b}\right) \tag{26}$$

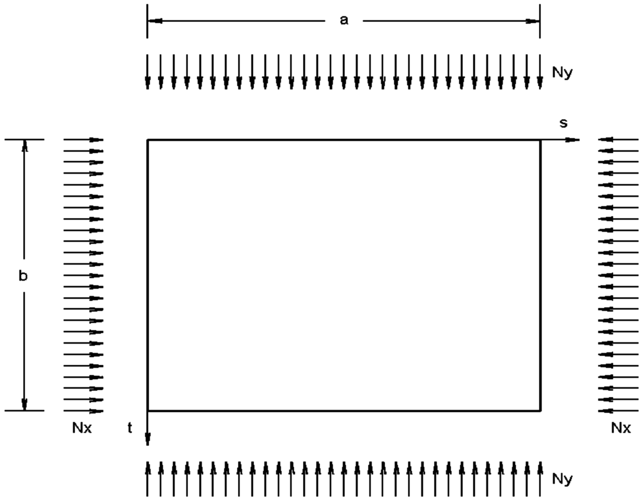

Fig. 9 Simply supported preloaded plate.

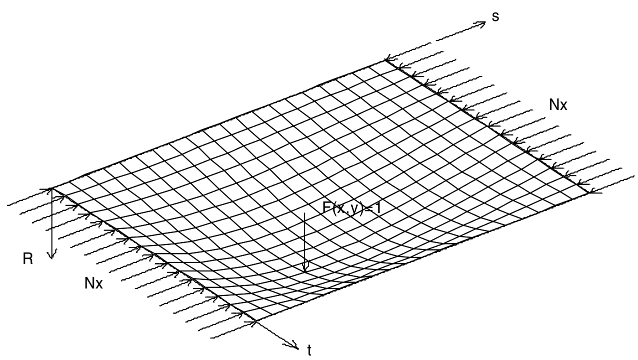

Fig. 10 Unit force on preloaded simply supported plate.

By substitution of Eq. (26) into Eq. (25), one obtains

$$b_{ij} = \frac{(4/ab)\sin(i\pi x/a)\sin(j\pi y/b)}{D[(i\pi/a)^2 + (j\pi/b)^2]^2 - N_x(i\pi/a)^2} \tag{27}$$

Using the now-known Green function R, one can obtain the additional load N_y, which will cause the plate to buckle. For definiteness, it will be assumed that [9]

$$\frac{1}{\sqrt{2}} < \frac{a}{b} < \sqrt{2} \tag{28}$$

In this case, the solution of Eq. (24) is the first buckling mode

$$w(x, y) = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) \tag{29}$$

Because of the orthogonal properties of buckling modes, Eq. (29) awakens, from the infinite series of R, only the first buckling mode. This explains why the approximation applied in example 3 yields the exact solution. As expected [9], the solution of Eq. (24) yields

$$N_y = N_{y\text{cr}}\left(1 - \frac{N_x}{N_{x\text{cr}}}\right) \tag{30}$$

where $N_{x\text{cr}}$ and $N_{y\text{cr}}$ are the critical loads when the plate is loaded, respectively, only in the x or y direction. It must be emphasized, however, that the preload, in general, can be also tension.

Conclusions

It was shown that the generic integral equations for the buckling of plates and bars, but not their derivatives, include the influence of the shear strain on the magnitude of the additional load that causes the preloaded structure to buckle.

References

[1] Baruch M., "Nondestructive Determination of Buckling for Plates and Bars Including Shear," *AIAA Journal*, Vol. 40, No. 11, 2002, pp. 2354–2359.
[2] Baruch, M., "Integral Equations for Nondestructive Determination of Buckling Loads for Elastic Plates and Bars," *Israel Journal of Technology*, Vol. 11, Nos. 1–2, 1973, pp. 1–8.
[3] Segall, A., and Baruch, M., "Nondestructive Dynamic Method for the Determination of the Critical Load of Elastic Columns," *Experimental Mechanics*, Vol. 20, No. 8, 1980, pp. 285–288. doi:10.1007/BF02328413
[4] Segall, A., and Springer, G. S., "Dynamic Method for Measuring the Critical Loads of Elastic Flat Plates," *Experimental Mechanics*, Vol. 26, No. 4, 1986, pp. 354–359. doi:10.1007/BF02320151
[5] Salunkhe, A. L., and Mujumdar, P. M., "Identification Approach to Estimate Buckling Load of Damaged Composite Plates," *AIAA Journal*, Vol. 36, No. 8, 1998, pp. 1479–1485.
[6] Baruch, M., "Nondestructive Determination of Additional Buckling Load for Preloaded Plates and Bars," Technical Aeronautics Rept. No. 296, Faculty of Aerospace Engineering, Technion–Israel Inst. of Technology, Haifa, Israel, Nov. 2003.
[7] Hoff, N. J., *Analysis of Structures*, Pt. 2, Wiley, New York, 1956.
[8] Sokolnikoff, I. S., *Mathematical Theory of Elasticity*, 2nd ed., McGraw–Hill, New York, 1956, p. 106.
[9] Timoshenko, S. P., *Theory of Elastic Stability*, 2nd ed., McGraw–Hill, New York, 1961, pp. 28, 132, 357.
[10] Kirchhoff, G., "Uber das Gleichgewicht und Bewegung einer elastichen Scheibe," *Journal für die Reine und Angewandte Mathematik*, Vol. 40, Jan. 1850, pp. 51–88.

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